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Numerical Computation of the Scattering Frequencies for Acoustic Wave Equations*

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We report on the computation of the scattering frequencies (resonances or poles of the S-matrix) due to a finite potential. We use a discretization of the time-dependent wave equation together with a judicious implementation of a method of Prony.

1. INTRODUCTION

The purpose of this paper is to report on the computation of the scattering frequencies due to waves scattered by a potential. Many acoustic and electromagnetic problems can be modeled by such waves, or by waves scattered by a conducting body. Examples include the medical applications of ultrasonics, and the detection of targets by radar. An impenetrable body can be regarded as a region where the potential is infinite. Scattering frequencies are also known as *complex resonances* or *poles of the scattering matrix*. Under appropriate conditions on the potential, they form a discrete set $\{\sigma_j\}$ in the complex plane with $\operatorname{Im} \sigma_j \to +\infty$.

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Consider a potential $q(x)$ where $x \in \mathbb{R}^3$. Look for solutions of the problem

$$\begin{aligned} -\Delta p + q(x)\, p(x) &= \sigma^2 p(x) \\ p(x) &\sim \exp(-i\sigma r)/r \qquad \text{as} \quad r = |x| \to \infty. \end{aligned} \tag{1}$$

If this problem has a nontrivial solution, we say that σ is a *scattering frequency.*

We know of two methods in the literature for finding these resonances. One is a method of the Rayleigh–Ritz type, well known in quantum chemistry [14]. Another, closely related, method consists of discretizing the elliptic equation (1) with an appropriate boundary condition on a large sphere and then computing the eigenvalues of the resulting matrix [13], but we are not aware of any numerical results using this method. Instead, we proceed as follows.

First we solve the *time-dependent* wave equation

$$u_{tt} - \Delta u + q(x)\, u(x, t) = 0 \qquad \text{for} \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R} \tag{2}$$

for any convenient (but appropriately nontrivial) initial data of compact support at $t = 0$. It is known [5] that the asymptotic expansion

$$u(x, t) \sim \sum_{j=1}^{\infty} c_j e^{i\sigma_j t} p_j(x) \tag{3}$$

is valid for fixed x as $t \to +\infty$ (except in the case of multiple σ_j when powers of t may also occur). Here σ_j is the jth scattering frequency and $p_j(x)$ is the corresponding eigenfunction as in (1) above, both of which depend only on the potential. Only the coefficients c_j depend on the initial data.

The second step of our method is to fix x and calculate the exponents in (3) from the discrete numerical solutions evaluated at equally spaced sample times. Thus with any fixed choice of x and with $t = t_0 + k\Delta T$, we denote $u(x, t_0 + k\Delta T) = f_k$ and $z_j = \exp(i\sigma_j \Delta T)$. Then the asymptotic expansion takes the form (with different c_j)

$$f_k \sim \sum_{j=1}^{\infty} c_j z_j^k. \tag{4}$$

Therefore the second step is to calculate the z_j and c_j from the f_k. Of course, only a finite number of terms, say n of them, can be calculated. We have already studied this delicate numerical problem in [12, 7]. Our solution is basically an implementation of a classical idea of Prony [9], which reduces the problem to solving two systems of n linear equations and to finding the zeros of a polynomial of degree n. As a check on the accuracy of our results, we repeat the procedure using different sets of sampling locations and different initial data.

Our method is quite flexible. It can be applied to other scattering problems such as (i) the wave equation in the exterior of an obstable with well-posed boundary conditions, (ii) potentials or obstacles which depend periodically on time, or (iii) electromagnetic waves in the presence of a conductor or dielectric. In Section 5 we

report on problem (ii). In the future we shall report on problems (i) and (iii). The second step, solving (4), is similar to the statistical parameter estimation problem known as the method of moments. The main limitation of our method derives from the ill-conditioned nature of that step. That in turn requires a very accurate solution of the differential equation, which is the first step. Solving a two- or three-dimensional wave equation to such accuracy requires a computer with a large memory. Therefore the present paper is limited to the case of spherically symmetric potentials. In our computations we used an IBM 3081 computer with double precision, so that the roundoff error was about 10^{-12}.

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2. Computational Method

2.1. *Finite Difference Scheme*

Assuming radial symmetry in both the potential and the solutions, our equation (2) can be written as

$$\begin{gathered} u_{tt} - u_{rr} - (2/r)\, u_r + q(r) u = 0, \qquad r > 0 \\ u_r(0, t) = 0. \end{gathered} \tag{5}$$

The transformation $v = ru$ eliminates the apparent singularity at $r = 0$:

$$\begin{gathered} v_{tt} - v_{rr} + q(r) v = 0, \qquad r > 0 \\ v(0, t) = 0. \end{gathered}$$

To this we append initial conditions $v(r, 0) = h_1(r)$, $v_t(r, 0) = h_2(r)$. We assume that $q(r)$, $h_1(r)$, and $h_2(r)$ have their supports in $0 \leqslant r \leqslant R$, so that $v(r, t)$ vanishes for $r \geqslant R + t$. Since we solve only up to a finite time $t \leqslant T$, we need not be concerned about boundary conditions at infinity. Let $r_j = j\,\Delta r$ and $t_n = n\,\Delta t$, where Δr and Δt are the spatial and temporal grid sizes. The finite difference scheme is

$$[v_j^{n+1} - 2v_j^n + v_j^{n-1}] - s[v_{j+1}^n - 2v_j^n + v_{j-1}^n] + (\Delta t)^2 q(r_j)\, v_j^n = 0, \tag{6}$$

where $s = (\Delta t/\Delta r)^2$. The initial data are

$$\begin{gathered} v_j^0 = h_1(r_j), \\ v_j^1 = (1 - s - \tfrac{1}{2}(\Delta t)^2 q(r_j))\, v_j^0 + (s/2)[v_{j+1}^0 + v_{j-1}^0] + (\Delta t)\, h_2(r_j). \end{gathered}$$

By Lax's equivalence theorem, a difference scheme is convergent if and only if it is consistent and stable. This scheme is convergent if $0 < s \leqslant 1$. We choose $s = 1$, in which case it is of second order (see [10]).

2.2. *Prony's Algorithm*

With a choice of spatial point x, a sample rate ΔT (usually larger than the finite-difference mesh size Δt), and an integer n (the "presumed" number of poles), the second part of our procedure is to solve a system

$$f_k = \sum_{j=1}^{n} c_j z_j^k. \tag{7}$$

Prony's algorithm is as follows:

(A) Solve the system

$$\sum_{k=0}^{n} \alpha_k f_{k+m} = 0 \qquad (m = 0, \ldots, M-1) \tag{8}$$

for $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n$, where $\alpha_n = 1$.

(B) Solve for the zeros of the polynomial

$$p(z) = \sum_{k=0}^{n} \alpha_k z^k. \tag{9}$$

These provide the scattering frequencies by use of the equation $z_j = \exp(i\sigma_j \Delta T)$.

(C) If the coefficients c_j are also desired, solve the system

$$\sum_{j=1}^{n} c_j z_j^k = f_k \qquad (k = 0, 1, 2, \ldots). \tag{10}$$

We usually take M to be considerably larger than n and (8) is then an overdetermined linear system. The number of data values f_k used is $M+n$.

When we adapt the algorithm to the discrete solution of the wave equation, how shall we choose the starting time t_0? That is, how shall we choose $f_0 = u(r_0, t_0)$? We choose it just beyond the reach of the characteristics which emanate from the support of the initial data. That is, we pick $t_0 \geqslant R + r_0$. This is reasonable from the point of view of the theory and its efficacy is corroborated by numerical examples.

We also need to choose the step size Δt of the finite difference scheme so that the total number of time steps is not too large. It is therefore convenient to choose the support R fairly small. This can always be accomplished by a change of scale ($x \to ax$, $t \to at$) in the independent variables x and t. As for the sample rate ΔT, we know it must be taken neither too large nor too small or else the linear system (8) will be too ill-conditioned.

2.3. *Implementation of Prony's Algorithm*

In step (A) above we must solve system (8). We write it in the form

$$A_n \beta_n = -a_{n+1}, \tag{11}$$

where the vectors a_μ are $a_\mu = [f_{\mu-1}, ..., f_{\mu+M-2}]^t$ for $1 \leqslant \mu \leqslant n+1$, the matrix is $A_n = [a_1, ..., a_n]$, and the vector β_n is $\beta_n = [\alpha_0, ..., \alpha_{n-1}]^t$. This system is solved using the generalized inverse as follows. The singular value decomposition of A is $A = UDV^t$, where U and V are orthogonal matrices and D is a diagonal matrix with diagonal entries $s_1, ..., s_\mu, 0, ..., 0$ with $s_i > 0$. Next we let

$$A^+ = V \cdot D_\delta^+ \cdot U^t, \tag{12}$$

where D_δ^+ is the diagonal matrix with entries $s_1^+, ..., s_\mu^+, 0, ..., 0$ and s_i^+ equals $1/s_i$ for $s_i \geqslant \delta$ and equals 0 for $s_i < \delta$, for an appropriately chosen δ. Then the approximate solution of system (11) which we use is $\beta_n = -A^+ \cdot a_{n+1}$. For the details of implementing this solution we refer to [12 or 7].

In step (B) above we must find the complex roots of a polynomial $p(z)$. We use Muller's method. Namely, let $z_1, ..., z_{m-1}$ be known roots and let

$$p_m(x) = p(x)/\{(x-z_1)\cdots(x-z_{m-1})\}.$$

If x_0, x_1, and x_2 are given, we successively compute $x_3, x_4, ...$ as follows. We let $\pi(x)$ be the second-order Lagrange interpolant of $p_m(x)$ through the points $(x_i, p_m(x_i))$, $(x_{i-1}, p_m(x_{i-1}))$, and $(x_{i-2}, p_m(x_{i-2}))$. Then we set x_{i+1} equal to the root of $\pi(x)$ with the larger denominator in magnitude. We set z_m equal to x_i for a sufficiently large i. We have modified Muller's procedure to improve its accuracy and save computation time. We set $e_m = |z_m(i) - z_m(i-1)|$, where $z_m(i)$ denotes the approximation to z_m in the last iteration. We reorder the roots $z_1, ..., z_n$ so that $e_1 < e_2 < \cdots < e_n$. We repeat the iteration a number of times with starting points $x_0 = z_m$, $x_1 = 1 + z_m$, $x_2 = -1 + z_m$, in each procedure of finding the mth root. For details we refer to [12 or 7].

2.4. *Reliability of the Computational Method*

Our basic test of reliability is to look for repetitions of the scattering frequencies as the parameters are changed. We change: (i) the presumed number n of poles, (ii) the sample locations, and (iii) the initial data $h_1(r)$ and $h_2(r)$. By the sample locations, we mean the fixed spatial point x, the sample rate ΔT and the starting time t_0. In general, we have found that fewer than n roots are accurately repeated; in fact, they begin to repeat only when the smallest singular value of A_n has the same order as the numerical error in solving the wave equation. A calculation is considered accurate insofar as its repeats under a variety of choices of (i), (ii), and (iii).

We have tested our method in two cases where the exact values of the scattering frequencies are known. For our first example we let $q(x) = 6$ for $r < 0.3$ and $q(x) = 0$ otherwise. In this case an exact equation for the scattering frequencies σ is $\tanh(0.3\beta) = -\beta/\lambda$, where $\lambda = i\sigma$ and $\beta = (\lambda^2 + 6)^{1/2}$ (see Eq. (15) in Section 4). From this equation we get the exact values of $\lambda = i\sigma$. Table I compares the exact values of the first six scattering frequencies with the values computed using finite differences and the Prony method. In the computation we chose step sizes

TABLE I

Exact λ	Computed λ
$-7.0996 \pm 8.2731\,i$	$-7.100 \pm 8.273\,i$
$-9.5214 \pm 19.549\,i$	$-9.522 \pm 19.55\,i$
$-10.866 \pm 30.355\,i$	$-10.87 + 30.36\,i$
$-11.815 \pm 41.019\,i$	$-11.82 \pm 41.03\,i$
$-12.551 \pm 51.619\,i$	$-12.57 \pm 51.67\,i$
$-13.153 \pm 62.183\,i$	$-13.20 \pm 62.23\,i$

$\Delta t = \Delta r = 0.005$, sample rate $\Delta T = 0.02$, and presumed number of roots $n = 25$ and 30 and 35. We also used ten different sets of data values (at $r = 0.05j$ where $j = 1, ..., 10$) and two different sets of initial data ($\mathscr{A}$ and $\mathscr{B}$) of supports $0 \leqslant r \leqslant R = 0.3$. To be specific, the data we used were

$$\text{Data } \mathscr{A}\text{: } u(r, 0) = 10^4 r(r-R)^2, \qquad u_t(r, 0) = r(R-r) \qquad \text{for} \quad r \leqslant R.$$

$$\text{Data } \mathscr{B}\text{: } u(r, 0) = 100(R-r)\sin(2\pi r/R), \qquad u_t(r, 0) = \sin(2\pi r/R) \qquad \text{for} \quad r \leqslant R.$$

All of these choices gave agreement up to about $O(10^{-3})$ in the first six scattering frequencies and provided the computed values in Table I.

For another example, we let $q(x) = 10$ for $0.2 < r < 0.3$ and $q(x) = 0$ otherwise. Then an exact equation for $\lambda = i\sigma$ is

$$[\beta + \lambda \tanh(\beta/10)]/[\lambda + \beta \tanh(\beta/10)] = -[\beta/\lambda] \tanh(\lambda/5), \tag{13}$$

where $\beta = (\lambda^2 + 10)^{1/2}$ (see Eq. (16) in Section 4). The results of the computation, which used the same parameters as in the first example, are again in excellent agreement with the exact values, as shown in Table II.

TABLE II

Exact λ	Computed λ
$-5.9547 \pm 7.9007\,i$	$-5.955 \pm 7.901\,i$
$-8.4566 \pm 19.848\,i$	$-8.457 \pm 19.85\,i$
$-10.234 \pm 30.433\,i$	$-10.23 \pm 30.43\,i$
$-10.878 \pm 40.884\,i$	$-10.87 \pm 40.88\,i$
$-11.600 \pm 51.776\,i$	$-11.58 \pm 51.78\,i$
$-12.442 \pm 62.220\,i$	$-12.40 \pm 62.23\,i$

3. Computational Results

In this section we exhibit the first few scattering frequencies for a variety of radial potentials. In the computations we generally took initial data $\mathscr{A}$, step sizes $\Delta t = \Delta r = 0.0025$, sample rate $\Delta T = 0.02$, and $M = 80 - n$ so that 80 data points were used in Prony's algorithm. The computations were repeated for $n = 20$, 30, and 35 and at the spatial locations $r = 0.025j$ ($j = 1, ..., 5$), a total of 15 repetitions of the algorithm. In Figs. 1–6 the results are presented graphically for the reader's convenience. Each graph represents the first quadrant, or the first and fourth quadrants, of the complex plane, which is sufficient since the scattering frequencies occur in pairs symmetric across the imaginary axis. In Figs. 1, 3, 5, and 6 each scattering frequency is indicated by a circle and, for a given potential, is connected to its neighboring scattering frequencies by line segments, for visual convenience. The potentials are graphed in the upper left-hand corner. In Fig. 6 the potential is $q(r) = A(r) \sin(2\pi r/0.3)$ for $0 < r < 0.3$ and $q(r)$ vanishes for $r > 0.3$.

The following general conjectures may be made based on this numerical evidence. It should be kept in mind that we are considering only radial potentials $q(r)$ and only those scattering frequencies σ_j which are excited by radial initial data.

(a) If $q(r) \geq 0$, then $\operatorname{Re} \sigma_j \neq 0$ and $\operatorname{Im} \sigma_j > 0$.

(b) If $q(r) \leq 0$, then there is a j such that $\operatorname{Re} \sigma_j = 0$. If $q(r)$ changes sign, this phenomenon depends on how negative $q(r)$ is. See Figs. 5 and 6.

(c) $\operatorname{Re}(\sigma_{j+1} - \sigma_j)$ is close to π/R, especially for large j, where R is the radius of the support of $q(r)$. In most of the above examples, $R = 0.3$ so that $\pi/R = 10.46$. For instance, in the case of Fig. 1 with $A = 6$, $\operatorname{Re}(\sigma_6 - \sigma_5) = 10.56$ (cf. Table I).

(d) $\operatorname{Im} \sigma_j$ decreases monotonically as the amplitude of $q(r)$ increases. See Figs. 1 and 3.

(e) The scattering frequencies depend continuously on the potential. This is clear from all of the figures. The only apparent exception is seen in Figs. 3 and 5, where the σ_j grow as $A \to 0$. (In fact they tend to infinity, the reason being that if $A = 0$ there is no scattering frequency at all.)

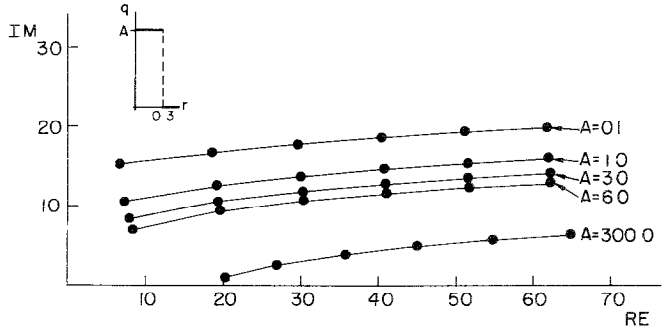

FIG. 1. Constant potential in a sphere.

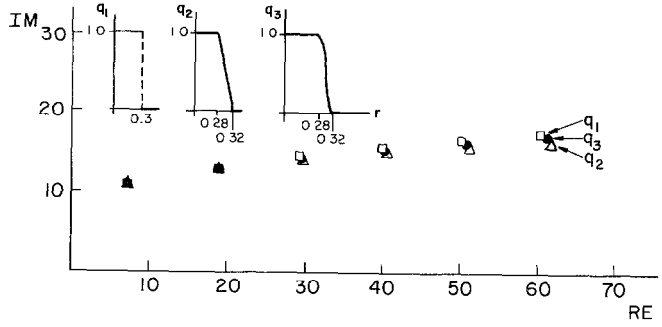

FIG. 2. Smoothed potential.

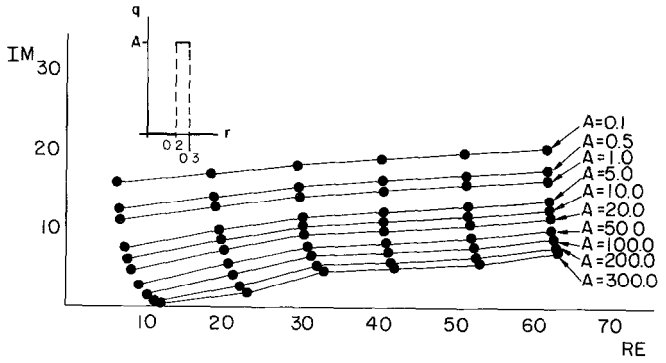

FIG. 3. Spherical shell.

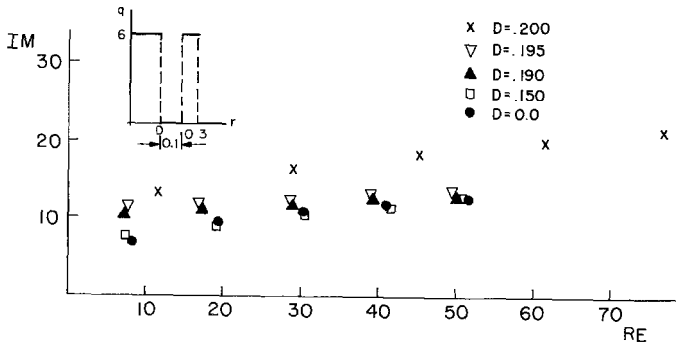

FIG. 4. Thin spherical shell.

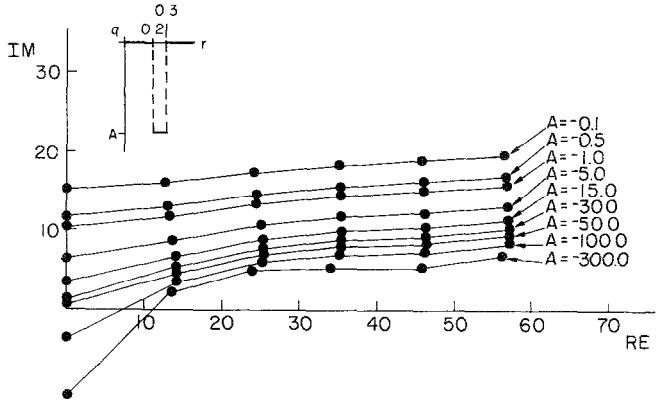

FIG. 5. Negative potential.

(f) The smoothness of $q(r)$ does not greatly affect the scattering frequencies but its influence does seem to increase with increasing j. See Fig. 2. (In order to distinguish the three potentials in Fig. 2, we took the finer mesh $\Delta t = \Delta r = 0.001$.)

(g) The scattering frequencies are relatively insensitive to the presence of "holes" (as in Fig. 4). Unless it is extremely thin, the outer "shell" of the support of $q(r)$ acts as a kind of barrier to the incident waves.

(h) If $q_A(r) = A \cdot q(r) \geqslant 0$, then $\operatorname{Im} \sigma_j \to 0$ as $A \to \infty$, while $\operatorname{Im} \sigma_j \to \infty$ as $A \to 0$. See Figs. 1 and 3. In Fig. 5 where the potential is negative, $\operatorname{Im} \sigma_j \to 0$ for $j \geqslant 2$, but $\sigma_1 \to -i\infty$ as $A \to -\infty$.

4. COMPARISON WITH THE THEORY

We begin with a brief summary of the theory of the scattering frequencies excited by a potential of compact support. The basic reference is [5]. Let $q(x)$ be a

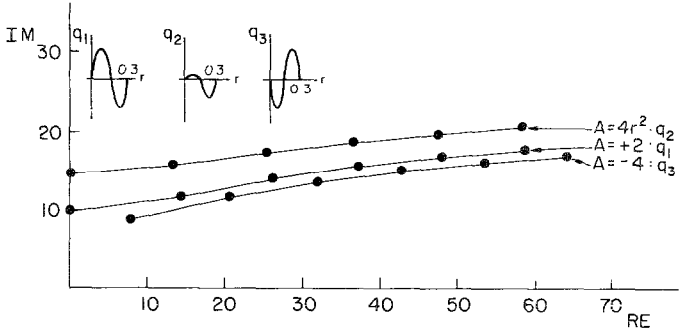

FIG. 6. Potentials which change sign.

bounded function for $x \in \mathbb{R}^3$ which vanishes for $r = |x| > R$. The first definition of a scattering frequency is a complex number σ for which there is a solution $p(x)$ (not identically zero) of

$$\begin{aligned} -\Delta p + q(x)\, p(x) &= \sigma^2 p(x) \\ p(x) &\sim \exp(-i\sigma r)/r \qquad \text{as} \quad r = |x| \to \infty. \end{aligned} \tag{1}$$

The condition at infinity is the classical outgoing radiation condition; it can also be expressed in a more abstract way. This definition defines σ as a kind of *eigenvalue*. Of course it is not a true eigenvalue because $p(x)$ is not square-integrable. In fact, most of the spectrum of the operator $-\Delta + q$ is continuous. A scattering frequency is also a number σ such that $\exp(i\sigma t)$ is a (true) eigenvalue of the Lax–Phillips semigroup $Z(t)$.

The second definition of a scattering frequency is as a *pole*. Solve the same equation as (1) but with the condition

$$p(x) - p_0(x) \sim \exp(-i\sigma r)/r \qquad \text{as} \quad r \to \infty,$$

where $p_0(x)$ is a solution of the free equation (with $q = 0$). If we call the solution $p(x, \sigma)$, then the scattering frequencies are the poles of the meromorphic function $p(x, \sigma)$ as a function of σ. An equivalent characterization, in case $q(x) \geqslant 0$, is as the poles of the resolvent operator $[\sigma^2 + \Delta - q]^{-1}$ when extended from the lower to the upper half plane. They are also the poles of the outgoing Green's function and of the S-matrix.

The third definition of the scattering frequencies is that they are the *exponents* which appear in the asymptotic expansion

$$u(x, t) \sim \sum_{j=1}^{\infty} c_j e^{i\sigma_j t} p_j(x) \tag{3}$$

as $t \to \infty$, where $u(x, t)$ is any nontrivial solution of

$$u_{tt} - \Delta u + q(x)\, u(x, t) = 0 \qquad \text{for} \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R} \tag{2}$$

with initial data of compact support.

The scattering frequencies depend only on the potential. The main difficulty, both theoretically and computationally, is that they cannot be characterized variationally. The following facts have been proved. They are confirmed in part by our computational results in Sectional 3.

(i) There are at most a countable number of scattering frequencies σ_j and $\operatorname{Im} \sigma_j \to +\infty$ [5].

(ii) If $q(x) \geqslant 0$, all the scattering frequencies are in the upper half plane [5]. More generally, the same is true if the operator $-\Delta + q$ has no negative eigenvalues. If this operator does have a negative eigenvalue $-\kappa^2$, then $-i\kappa$ is a scattering frequency. It corresponds to an exponentially growing mode; all the scattering frequencies in the lower half plane arise in this manner.

(iii) If $q(x) \geqslant 0$, there exists an infinite sequence of *imaginary* scattering frequencies [6]. We do not see them in our computations because we are only dealing with radial solutions.

(iv) If $q(x)$ is C^∞ then the number of scattering frequencies in a large disk $|\sigma| < N$ grows at most polynomially as $N \to \infty$ [8]. If $q(x)$ is even smoother, namely, if it is in the Gevrey class of order α, then all the scattering frequencies lie above an algebraic curve $y = c + c|x|^\beta$ in the complex plane where $\beta = 1/(3\alpha - 1)$ [3]. Thus the smoothness of the potential affects the asymptotic location of the scattering frequencies, in general agreement with our conjecture (f) in Section 3.

(v) Each scattering frequency depends continuously on $q(x)$ in the L^p norm for any $p > 3$, except if they tend to infinity [2].

(vi) In case the potential $q(r)$ depends only on r, we may look for solutions $p(r)$ of Eq. (1) which depend only on r. Then there are other solutions of the form $p(r)\, Y(\theta, \varphi)$, where $Y(\theta, \varphi)$ is any spherical harmonic. We did not see them in the computations because we were dealing only with radial solutions. Thus there is no contradiction between conjecture (a) above and Theorem (iii): all of the imaginary scattering frequencies in (iii) correspond to *non*-radial solutions. Conjectures (a) and (b) were suggested by the computations and we have subsequently been able to prove them [11].

(vii) We mention a theorem for the case of a bounded obstacle with Dirichlet boundary conditions instead of a potential [4]. If the obstacle consists of two disjoint smooth convex pieces a distance D apart, then there exists a sequence of scattering frequencies asymptotically close to $Ci + j\pi/D$ for $j = 1, 2, \ldots$. What the spacing π/D of the real parts represents is the frequency of a wave traveling along the ray joining the two pieces and back again, a distance $2D$. Our conjecture (c) above has a similar interpretation: the spacing π/R is the frequency of a spherical wave shrinking from the edge $r = R$ of the support of $q(x)$ to the origin and expanding back to the edge, a radial distance $2R$.

(viii) If $\mathcal{O}$ is a bounded domain and we consider a potential $Aq(x)$, where $q(x) > 0$ in $\mathcal{O}$, then the scattering frequencies of the potential converge, as $A \to \infty$, to the scattering frequencies of $\mathcal{O}$ and to the eigenvalues of the interior holes of $\mathcal{O}$ if $\mathcal{O}$ is not simply connected [6]. This is observed in Fig. 3, where the hole is the ball of radius 0.2 which has the lowest eigenvalue $\pi/0.2 = 15.7$.

We now derive explicit formulas for the radial scattering frequencies in case $q(r)$ is a step function. We begin with the simplest case, $q(r) = A$ for $r < R$, where A is a positive constant, and $q(r) = 0$ otherwise. Then $\lambda = i\sigma$ has negative real part and the problem (1) reduces to

$$\begin{aligned} p_{rr} + (2/r)\, p_r - [\lambda^2 + A]\, p = 0 \qquad & \text{for} \quad 0 < r < R, \\ p_r(0) = 0, \qquad & \\ p(r) = r^{-1} \exp(-\lambda r) \qquad & \text{for} \quad r > R. \end{aligned} \tag{14}$$

This is essentially Bessel's equation of order $\frac{1}{2}$ so that the solution which satisfies the boundary condition at $r=0$ is $r^{-1}\sinh(\beta r)$, where $\beta=(\lambda^2+A)^{1/2}$. Using the boundary condition at $r=R$, we easily obtain the equation

$$\tanh(\beta R)=-\beta/\lambda \tag{15}$$

for the radial scattering frequencies.

Next we take the potential $q(r)=A$ for $R_1<r<R_2$ and $q(r)=0$ otherwise. In this case $p(r)=r^{-1}\exp(-\lambda r)$ for $r>R_2$, $p(r)=Br^{-1}\sinh(\lambda r)$ for $r<R_1$, and $p(r)=r^{-1}[C\sinh(\beta r)+D\cosh(\beta r)]$ for $R_1<r<R_2$. This leads to the exact equation

$$\lambda\{\beta+\lambda\tanh[\beta(R_2-R_1)]\}/\{\lambda+\beta\tanh[\beta(R_2-R_1)]\}=-\beta\tanh[\lambda R_1] \tag{16}$$

for the radial scattering frequencies.

5. Time-dependent Potentials

In this section we present the extension of the theory and the computations to a potential which depends periodically on time. The basic reference is [1]. Let $q(x, t)$ be a bounded function which vanishes for $|x|>R$ and is periodic in time: $q(x, t+T)=q(x, t)$ for all x, t. Once again there are three basic definitions of a scattering frequency. First, it is a complex number σ for which there is a periodic function $p(x, t)$ of period T such that $\exp(i\sigma t)\,p(x, t)$ is an outgoing solution of the time-dependent equation

$$u_{tt}-\Delta u+q(x, t)\,u(x, t)=0. \tag{17}$$

"Outgoing" means that the solution u at any point (x^*, t^*) can be expressed as an integral of qu over the past cone $\{(x, t)\colon |x-x^*|=t^*-t\}$. Equivalently, if the perturbation q is suddenly removed, the solution in the future vanishes in an expanding sphere.

The second definition is as a *pole*. Solve the inhomogeneous equation (17) with any initial conditions but with the extra forcing term $\exp(i\sigma t)f(x, t)$, where $f(x, t)$ is any nontrivial function of period T. Then the solution will be meromorphic in σ with its poles at the scattering frequencies. There are also the poles of the S-matrix or scattering amplitude.

The third definition is as the *exponents* σ_j in the asymptotic expansion

$$u(x, t)\sim\sum_{j=1}^{\infty}c_j e^{i\sigma_j t}p_j(x, t) \tag{18}$$

as $t\to+\infty$, where $u(x, t)$ is any nontrivial solution of (17) and the $p_j(x, t)$ have period T and the other properties stated above.

The scattering frequencies depend only on the potential $q(x, t)$. Since any integer multiple of ν. where $\nu = 2\pi/T$ is the frequency of the potential, can be freely added to a scattering frequency σ to get another one, we normalize the definition to require $0 \leqslant \operatorname{Re} \sigma < \nu$. The following facts have been proved.

(α) There are at most a countable number of scattering frequencies σ_j and $\operatorname{Im} \sigma_j \to +\infty$ [2].

(β) Each scattering frequency depends continuously on q in the uniform (L^∞) norm [2].

In this paper we only consider radial potentials $q(r, t)$ and radial solutions $u(r, t)$ for the reasons mentioned in Section 1. Then Eq. (17) can be written in the form (5) except that $q = q(r, t)$. The finite difference scheme and Prony's algorithm are the same as in Section 2. We take the sample rate ΔT to be a multiple of the period T, so that $p(x, t_0 + k\,\Delta T) = p(x, t_0)$.

For our first computational example we chose

$$q(r, t) = 6 + 6B \sin(200\pi t) \qquad \text{for} \quad 0 \leqslant r < 0.3 \tag{19}$$

and $q(r, t) = 0$ otherwise, where $B = 0$, 0.01, 0.05, 0.1, and 0.5. This is the potential of Fig. 1 but with a rapidly oscillating amplitude. The scattering frequencies turned out to be very insensitive to B (within 1%).

A more interesting potential is one whose support is oscillating. We took

$$q(r, t) = 6 \qquad \text{for} \quad 0 \leqslant r < 0.3\{1 + D \sin(100\pi t)\} \tag{20}$$

and $q(r, t) = 0$ otherwise. The case $D = 0$ corresponds to no oscillation at all. The scattering frequencies are presented in Fig. 7 for $D = 0.1$, 0.3, and 0.5. We took our usual sample rate $\Delta T = 0.02$, which was the same as the period $T = 0.02$ of the potential, so that Prony's method could not "see" the scattering frequencies $\sigma + \nu$, $\sigma + 2\nu$, ... (The finite difference scheme, on the other hand, did "see" the periodicity since the mesh size was much smaller.)

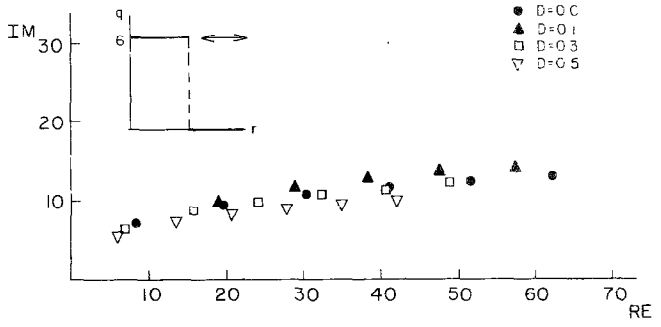

FIG. 7. Rapidly oscillating sphere.

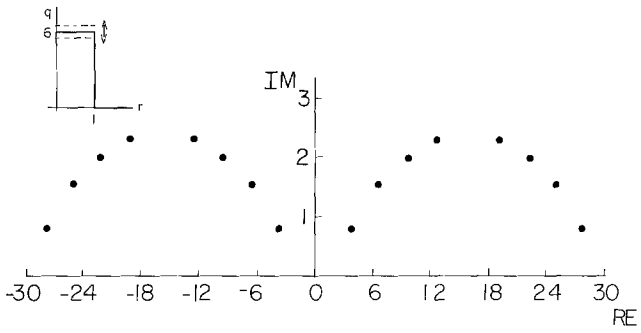

FIG. 8. Oscillating amplitude.

Next we took a potential with a smaller period, namely,

$$q(r, t) = 6 + 0.6 \sin(vt) \qquad \text{for} \quad 0 \leqslant r < 1 \tag{21}$$

and $q(r, t) = 0$ otherwise. The results are presented in Fig. 8 for $v = 10\pi$, using the sample rate $\Delta T = 0.1$. Thus Prony's method samples two data points per period ($T = 2\pi/v = 0.2$) and we are able to see scattering frequencies of the form $\sigma = \alpha + i\beta$, $-\alpha + i\beta$, $-v + \alpha + i\beta$, and $v - \alpha + i\beta$, two of which lie in the first quadrant and two in the second quadrant. In order to clearly illustrate this repetition of scattering frequencies, we graph the first four such quadruples in the upper half plane in Fig. 8. Note the pairs which are horizontally spaced apart exactly at the distance $v = 2\pi/0.2 = 10\pi = 31.4\ldots$.

We would like to study potentials which oscillate much more slowly, but such a calculation would require far slower sampling and therefore a much longer time to accurately solve the wave equation.

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